Differential Equation

Tuesday, 25 October 2022 8:25 AM

O/N/2012/Q6

The variables x and y are related by the differential equation

$$x\frac{dy}{dx} = 1 - y^2.$$

When $x = 2$, $y = 0$. Solve the differential equation, obtaining an expression for y in terms of x. [8]

O/N/2006/Q4

Given that $y = 2$ when $x = 0$, solve the differential equation

$$y\frac{dy}{dx} = 1 + y^2,$$

obtaining an expression for y^2 in terms of x. [6]

(i) Using partial fractions, find

$$\int \frac{1}{y(4-y)}\,\mathrm{d}y.$$ [4]

(ii) Given that $y = 1$ when $x = 0$, solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y(4-y),$$

obtaining an expression for y in terms of x. [4]

(iii) State what happens to the value of y if x becomes very large and positive. [1]

M/J/2009/Q8

(i) Express $\dfrac{100}{x^2(10-x)}$ in partial fractions. [4]

(ii) Given that $x = 1$ when $t = 0$, solve the differential equation

$$\frac{dx}{dt} = \frac{1}{100}x^2(10-x),$$

obtaining an expression for t in terms of x. [6]

(i) Express $\dfrac{1}{x^2(2x+1)}$ in the form $\dfrac{A}{x^2} + \dfrac{B}{x} + \dfrac{C}{2x+1}$. [4]

(ii) The variables x and y satisfy the differential equation

$$y = x^2(2x+1)\frac{dy}{dx},$$

and $y = 1$ when $x = 1$. Solve the differential equation and find the exact value of y when $x = 2$. Give your value of y in a form not involving logarithms. [7]

O/N/2017/Q5

The variables x and y satisfy the differential equation

$$(x+1)\frac{dy}{dx} = y(x+2),$$

and it is given that $y = 2$ when $x = 1$. Solve the differential equation and obtain an expression for y in terms of x. [7]

M/J/2012/Q5

The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = e^{2x+y},$$

and $y = 0$ when $x = 0$. Solve the differential equation, obtaining an expression for y in terms of x. [6]

M/J/2019/Q7

The variables x and y satisfy the differential equation $\dfrac{dy}{dx} = xe^{x+y}$. It is given that $y = 0$ when $x = 0$.

(i) Solve the differential equation, obtaining y in terms of x. [7]

(ii) Explain why x can only take values that are less than 1. [1]

O/N/2011/Q4

The variables x and θ are related by the differential equation

$$\sin 2\theta \frac{\mathrm{d}x}{\mathrm{d}\theta} = (x + 1)\cos 2\theta,$$

where $0 < \theta < \frac{1}{2}\pi$. When $\theta = \frac{1}{12}\pi$, $x = 0$. Solve the differential equation, obtaining an expression for x in terms of θ, and simplifying your answer as far as possible. [7]

O/N/2019/Q6

The variables x and θ satisfy the differential equation

$$\sin \tfrac{1}{2}\theta \frac{\mathrm{d}x}{\mathrm{d}\theta} = (x+2)\cos \tfrac{1}{2}\theta$$

for $0 < \theta < \pi$. It is given that $x = 1$ when $\theta = \frac{1}{3}\pi$. Solve the differential equation and obtain an expression for x in terms of $\cos\theta$. [8]

M/J/2010/Q7

The variables x and t are related by the differential equation

$$e^{2t}\frac{dx}{dt} = \cos^2 x,$$

where $t \geqslant 0$. When $t = 0$, $x = 0$.

(i) Solve the differential equation, obtaining an expression for x in terms of t. [6]

(ii) State what happens to the value of x when t becomes very large. [1]

(iii) Explain why x increases as t increases. [1]

In a certain industrial process, a substance is being produced in a container. The mass of the substance in the container t minutes after the start of the process is x grams. At any time, the rate of formation of the substance is proportional to its mass. Also, throughout the process, the substance is removed from the container at a constant rate of 25 grams per minute. When $t = 0$, $x = 1000$ and $\frac{dx}{dt} = 75$.

(i) Show that x and t satisfy the differential equation

$$\frac{dx}{dt} = 0.1(x - 250).$$ [2]

(ii) Solve this differential equation, obtaining an expression for x in terms of t. [6]

O/N/2010/Q10

A certain substance is formed in a chemical reaction. The mass of substance formed t seconds after the start of the reaction is x grams. At any time the rate of formation of the substance is proportional to $(20 - x)$. When $t = 0$, $x = 0$ and $\frac{dx}{dt} = 1$.

(i) Show that x and t satisfy the differential equation

$$\frac{dx}{dt} = 0.05(20 - x).$$ [2]

(ii) Find, in any form, the solution of this differential equation. [5]

(iii) Find x when $t = 10$, giving your answer correct to 1 decimal place. [2]

(iv) State what happens to the value of x as t becomes very large. [1]

O/N/2005/Q8

In a certain chemical reaction the amount, x grams, of a substance present is decreasing. The rate of decrease of x is proportional to the product of x and the time, t seconds, since the start of the reaction. Thus x and t satisfy the differential equation

$$\frac{dx}{dt} = -kxt,$$

where k is a positive constant. At the start of the reaction, when $t = 0$, $x = 100$.

(i) Solve this differential equation, obtaining a relation between x, k and t. [5]

(ii) 20 seconds after the start of the reaction the amount of substance present is 90 grams. Find the time after the start of the reaction at which the amount of substance present is 50 grams. [3]

A model for the height, h metres, of a certain type of tree at time t years after being planted assumes that, while the tree is growing, the rate of increase in height is proportional to $(9-h)^{\frac{1}{3}}$. It is given that, when $t=0$, $h=1$ and $\frac{dh}{dt}=0.2$.

(i) Show that h and t satisfy the differential equation

$$\frac{dh}{dt}=0.1(9-h)^{\frac{1}{3}}.$$ [2]

(ii) Solve this differential equation, and obtain an expression for h in terms of t. [7]

(iii) Find the maximum height of the tree and the time taken to reach this height after planting. [2]

(iv) Calculate the time taken to reach half the maximum height. [1]

O/N/2007/Q7

The number of insects in a population t days after the start of observations is denoted by N. The variation in the number of insects is modelled by a differential equation of the form

$$\frac{dN}{dt} = kN\cos(0.02t),$$

where k is a constant and N is taken to be a continuous variable. It is given that $N = 125$ when $t = 0$.

(i) Solve the differential equation, obtaining a relation between N, k and t. [5]

(ii) Given also that $N = 166$ when $t = 30$, find the value of k. [2]

(iii) Obtain an expression for N in terms of t, and find the least value of N predicted by this model. [3]

O/N/2009/Q9

The temperature of a quantity of liquid at time t is θ. The liquid is cooling in an atmosphere whose temperature is constant and equal to A. The rate of decrease of θ is proportional to the temperature difference $(\theta - A)$. Thus θ and t satisfy the differential equation

$$\frac{d\theta}{dt} = -k(\theta - A),$$

where k is a positive constant.

(i) Find, in any form, the solution of this differential equation, given that $\theta = 4A$ when $t = 0$. [5]

(ii) Given also that $\theta = 3A$ when $t = 1$, show that $k = \ln\frac{3}{2}$. [1]

(iii) Find θ in terms of A when $t = 2$, expressing your answer in its simplest form. [3]

The population of a country at time t years is N millions. At any time, N is assumed to increase at a rate proportional to the product of N and $(1 - 0.01N)$. When $t = 0$, $N = 20$ and $\frac{dN}{dt} = 0.32$.

(i) Treating N and t as continuous variables, show that they satisfy the differential equation

$$\frac{dN}{dt} = 0.02N(1 - 0.01N).$$ [1]

(ii) Solve the differential equation, obtaining an expression for t in terms of N. [8]

(iii) Find the time at which the population will be double its value at $t = 0$. [1]

O/N/2014/Q7

In a certain country the government charges tax on each litre of petrol sold to motorists. The revenue per year is R million dollars when the rate of tax is x dollars per litre. The variation of R with x is modelled by the differential equation

$$\frac{dR}{dx} = R\left(\frac{1}{x} - 0.57\right),$$

where R and x are taken to be continuous variables. When $x = 0.5$, $R = 16.8$.

(i) Solve the differential equation and obtain an expression for R in terms of x. [6]

(ii) This model predicts that R cannot exceed a certain amount. Find this maximum value of R. [3]

M/J/2015/Q9

The number of organisms in a population at time t is denoted by x. Treating x as a continuous variable, the differential equation satisfied by x and t is

$$\frac{dx}{dt} = \frac{xe^{-t}}{k + e^{-t}},$$

where k is a positive constant.

(i) Given that $x = 10$ when $t = 0$, solve the differential equation, obtaining a relation between x, k and t. [6]

(ii) Given also that $x = 20$ when $t = 1$, show that $k = 1 - \frac{2}{e}$. [2]

(iii) Show that the number of organisms never reaches 48, however large t becomes. [2]

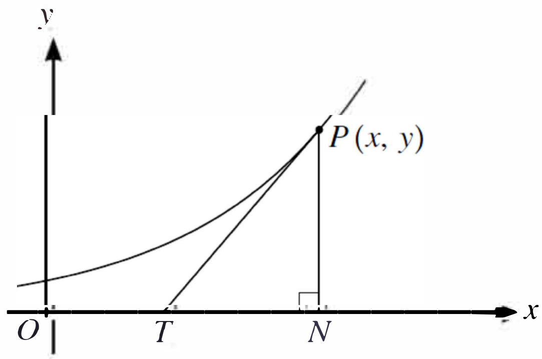
In a certain chemical process a substance A reacts with and reduces a substance B. The masses of A and B at time t after the start of the process are x and y respectively. It is given that $\frac{dy}{dt} = -0.2xy$ and $x = \frac{10}{(1+t)^2}$. At the beginning of the process $y = 100$.

(i) Form a differential equation in y and t, and solve this differential equation. [6]

(ii) Find the exact value approached by the mass of B as t becomes large. State what happens to the mass of A as t becomes large. [2]

O/N/2018/Q6

A certain curve is such that its gradient at a general point with coordinates (x, y) is proportional to $\frac{y^2}{x}$. The curve passes through the points with coordinates $(1, 1)$ and $(e, 2)$. By setting up and solving a differential equation, find the equation of the curve, expressing y in terms of x. [8]



In the diagram, the tangent to a curve at the point P with coordinates (x, y) meets the x-axis at T. The point N is the foot of the perpendicular from P to the x-axis. The curve is such that, for all values of x, the gradient of the curve is positive and $TN = 2$.

(i) Show that the differential equation satisfied by x and y is $\dfrac{dy}{dx} = \frac{1}{2}y$. [1]

The point with coordinates $(4, 3)$ lies on the curve.

(ii) Solve the differential equation to obtain the equation of the curve, expressing y in terms of x. [5]

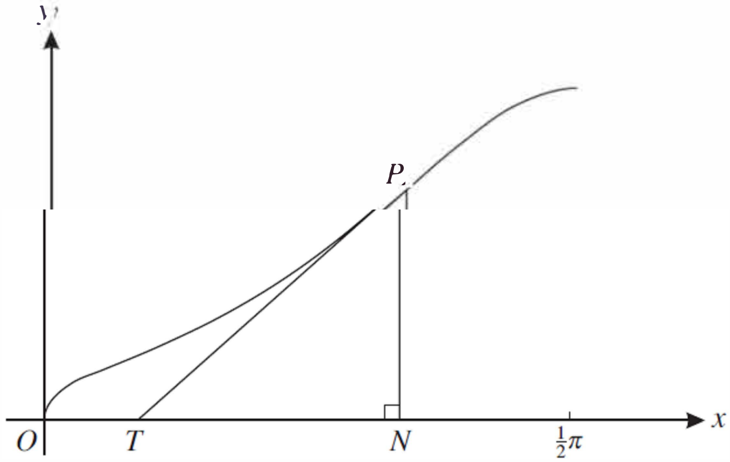
M/J/2011/Q6

A certain curve is such that its gradient at a point (x, y) is proportional to xy. At the point $(1, 2)$ the gradient is 4.

(i) By setting up and solving a differential equation, show that the equation of the curve is $y = 2e^{x^2-1}$. [7]

(ii) State the gradient of the curve at the point $(-1, 2)$ and sketch the curve. [2]

M/J/2008/Q8



In the diagram the tangent to a curve at a general point P with coordinates (x, y) meets the x-axis at T. The point N on the x-axis is such that PN is perpendicular to the x-axis. The curve is such that, for all values of x in the interval $0 < x < \frac{1}{2}\pi$, the area of triangle PTN is equal to $\tan x$, where x is in radians.

(i) Using the fact that the gradient of the curve at P is $\frac{PN}{TN}$, show that

$$\frac{dy}{dx} = \frac{1}{2}y^2 \cot x.$$ [3]

(ii) Given that $y = 2$ when $x = \frac{1}{6}\pi$, solve this differential equation to find the equation of the curve, expressing y in terms of x. [6]

A rectangular reservoir has a horizontal base of area $1000\,\mathrm{m}^2$. At time $t = 0$, it is empty and water begins to flow into it at a constant rate of $30\,\mathrm{m}^3\,\mathrm{s}^{-1}$. At the same time, water begins to flow out at a rate proportional to $\sqrt{h}$, where h m is the depth of the water at time t s. When $h = 1$, $\frac{dh}{dt} = 0.02$.

(i) Show that h satisfies the differential equation

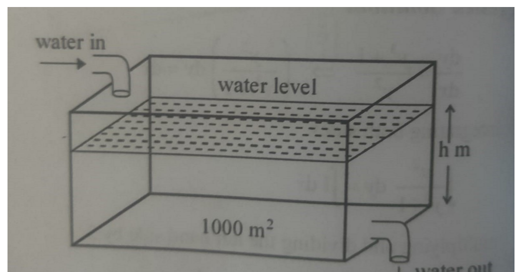
$$\frac{dh}{dt} = 0.01(3 - \sqrt{h}).$$ [3]

It is given that, after making the substitution $x = 3 - \sqrt{h}$, the equation in part **(i)** becomes

$$(x - 3)\frac{dx}{dt} = 0.005x.$$

(ii) Using the fact that $x = 3$ when $t = 0$, solve this differential equation, obtaining an expression for t in terms of x. [5]

(iii) Find the time at which the depth of water reaches 4 m. [2]



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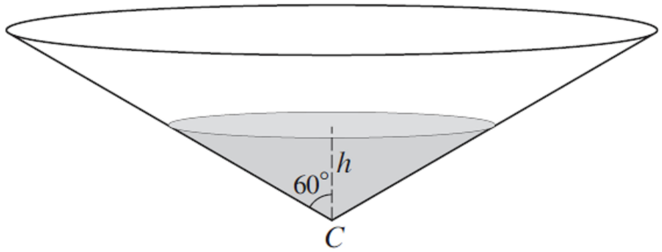


A tank containing water is in the form of a cone with vertex C. The axis is vertical and the semi-vertical angle is $60°$, as shown in the diagram. At time $t = 0$, the tank is full and the depth of water is H. At this instant, a tap at C is opened and water begins to flow out. The volume of water in the tank decreases at a rate proportional to $\sqrt{h}$, where h is the depth of water at time t. The tank becomes empty when $t = 60$.

(i) Show that h and t satisfy a differential equation of the form

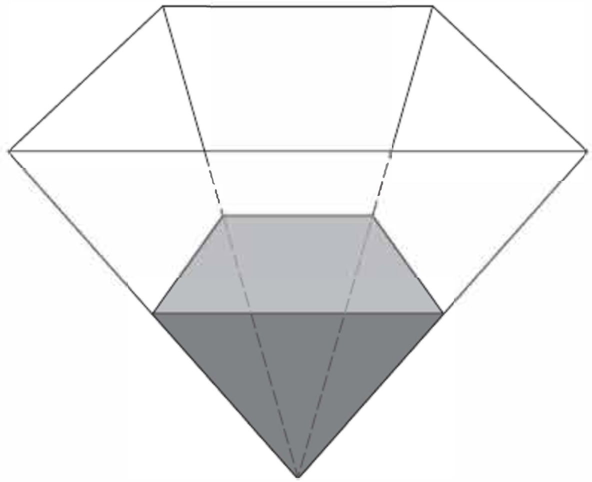
$$\frac{dh}{dt} = -Ah^{-\frac{3}{2}},$$

where A is a positive constant. [4]

(ii) Solve the differential equation given in part **(i)** and obtain an expression for t in terms of h and H. [6]

(iii) Find the time at which the depth reaches $\frac{1}{2}H$. [1]

[The volume V of a cone of vertical height h and base radius r is given by $V = \frac{1}{3}\pi r^2 h$.]



An underground storage tank is being filled with liquid as shown in the diagram. Initially the tank is empty. At time t hours after filling begins, the volume of liquid is V m^3 and the depth of liquid is h m. It is given that $V = \frac{4}{3}h^3$.

The liquid is poured in at a rate of 20 m^3 per hour, but owing to leakage, liquid is lost at a rate proportional to h^2. When $h = 1$, $\frac{dh}{dt} = 4.95$.

(i) Show that h satisfies the differential equation

$$\frac{dh}{dt} = \frac{5}{h^2} - \frac{1}{20}.$$ [4]

(ii) Verify that $\frac{20h^2}{100 - h^2} \equiv -20 + \frac{2000}{(10-h)(10+h)}$. [1]

(iii) Hence solve the differential equation in part **(i)**, obtaining an expression for t in terms of h. [5]